Deformed SU(2) Heisenberg chain

1991 J. Phys. A: Math. Gen. 24 581

(http://iopscience.iop.org/0305-4470/24/3/016)

J. Phys. A: Math. Gen. 24 (1991) 581-591. Printed in the UK

Deformed SU(2) Heisenberg chain

S Meljanac†, M Milekovi醇 and S Pallua‡

† Ruder Bošković Institute, 41001 Zagreb, Bijenička c 54, Croatia, Yugoslavia
‡ Prirodoslovno-Matematički Fakultet, Department of Theoretical Physics, 41000 Zagreb, Marulićev trg 19, Croatia, Yugoslavia

Received 6 August 1990

Abstract. The general Hamiltonian for the $SU(2)_q$-invariant arbitrary-spin Heisenberg chain is presented. Some of these interactions are shown to satisfy braid group relations and the Temperley-Lieb algebra relations. Spin $\frac{3}{2}$ is explored in more detail and, in this case, a general solution to the braid conditions is known in the generic sense.

1. Introduction

In recent years much progress has been made in the construction of integrable models of statistical physics in two dimensions and spin chains in one dimension [1, 2]. An essential tool in the search for such models was the construction of representations of braid groups, the Hecke and the Temperley-Lieb algebras in terms of observables of the system [3]. An interesting class of models can be obtained with q-deformations [4] of the SU(2)-invariant spin chain. Some properties of the spin-1 model have been studied by Batchelor *et al* [5]. In this paper we shall study the SU(2) deformed chain for arbitrary spin. We shall study spin $\frac{3}{2}$ in more detail.

We recall the defining relations for the SU(2) deformed algebra (hereafter referred to as $SU(2)_q$). The generators J^+, J^- and J^3 satisfy

$$[J^3, J^\pm] = \pm J^\pm \qquad [J^+, J^-] = [2J^3] \tag{1.1}$$

where

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{1.2}$$

In the tensor product $V_k \times V_{k+1}$ of spaces V_k and V_{k+1} we can define the coproduct

$$\begin{aligned}\Delta_{k,k+1}(J^\pm) &= q^{-J^3_k} \otimes J^\pm_{k+1} + J^\pm_k \otimes q^{J^3_{k+1}} \\ \Delta_{k,k+1}(q^{\pm J^3}) &= q^{\pm J^3_k} \otimes q^{\pm J^3_{k+1}}.\end{aligned} \tag{1.3}$$

The Casimir operator is given by

$$C = J^- J^+ + [J^3 + \tfrac{1}{2}]^2 - [\tfrac{1}{2}]^2. \tag{1.4}$$

Its eigenvalue c_j in the irreducible representation of spin j is

$$c_j = [j + \tfrac{1}{2}]^2 - [\tfrac{1}{2}]^2 = [j][j+1]. \tag{1.5}$$

We introduce the Hamiltonian for the $SU(2)_q$ spin chain of $n-1$ sites

$$H = \sum_{k=1}^{n-1} I_k \tag{1.6}$$

where I_k is the most general polynomial function of the Casimir operator $\Delta_{k,k+1}(C)$ defined on the vector space $V_k \times V_{k+1}$. Since Δ is a homomorphism, we can write

$$\Delta_{k,k+1}(C) = \Delta_{k,k+1}(J^-)\Delta_{k,k+1}(J^+) + (\Delta_{k,k+1}([J^3+\tfrac{1}{2}])^2 - [\tfrac{1}{2}]^2 1 \times 1). \tag{1.7}$$

We assume that site $n-1$ is followed by site 1, so that J_n in (1.7) is to be interpreted as J_1.

For simplicity, from now on we shall often denote

$$I_k = I \qquad \Delta_{k,k-1}(C) = C. \tag{1.8}$$

It is well known [4] that irreducible representations of $SU(2)_q$ are classified in the same way as those of SU(2). This means that we have a $(2j+1)$-dimensional representation with $j = 0, \frac{1}{2}, \ldots$ with a possible basis

$$\begin{aligned} &|jm\rangle \qquad m = j, j-1, \ldots, -j+1, -j \\ &J^{\pm}|jm\rangle = \sqrt{[j \mp m][j \pm m + 1]}\,|jm \pm 1\rangle \\ &J^3|jm\rangle = m|jm\rangle. \end{aligned} \tag{1.9}$$

Thus in the tensor product of spaces $V_1 \times V_2$ we can choose the basis

$$|j_1 m_1 j_2 m_2\rangle = |j_1 m_1\rangle \times |j_2 m_2\rangle. \tag{1.10}$$

This tensor product can be decomposed into its irreducible components denoted by

$$J = j_1 + j_2, \ldots, |j_1 - j_2|.$$

In each component we can form eigenstates of J^3

$$J^3|JMj_1j_2\rangle = M|JMj_1j_2\rangle \qquad M = J, J-1, \ldots, -J. \tag{1.11}$$

The connection between the two basis is given in terms of Clebsch–Gordan coefficients

$$|JMj_1j_2\rangle = \sum_{m_1,m_2} C^{JM}_{j_1m_1j_2m_2}|j_1m_1\rangle|j_2m_2\rangle. \tag{1.12}$$

The explicit expressions for Clebsch–Gordan coefficients are known [4, 6]. From now on we shall treat interactions of the same spin

$$j_1 = j_2 = j.$$

This means that because of (1.5) the Casimir operator on the space $V_k \times V_{k+1}$ will have the following eigenvalues:

$$c_J = [J][J+1] \qquad J = 0, 1, \ldots, 2j \tag{1.13}$$

or

$$c_0 = 0 \qquad c_1 = [2] \qquad c_2 = [2][3], \ldots, c_{2j} = [2j][2j+1].$$

2. $SU(2)_q$-invariant interaction

Now we are able to write the most general local interaction I. Linearly independent contributions are given by

$$1, C, C^2, \ldots, C^{2j}.$$

This series terminates because of the vanishing of the characteristic polynomial

$$(C - c_0)(C - c_1), \ldots, (C - c_{2j}) = 0. \tag{2.1}$$

In other words, C^{2j+1} is a linear combination of 1×1, C, $C^2 \dots C^{2j}$. For convenience, we write the most general interaction in the following polynomial form:

$$I = \alpha_0 1 + \alpha_1 (C - c_{2j}) + \alpha_2 (C - c_2)(C - c_{2j} - 1) \dots + \alpha_{2j}(C - c_{2j}) \dots (C - c_1). \tag{2.2}$$

Sometimes it may be useful to use another representation for the interaction where projector operators are chosen for the basis. More precisely, we introduce projectors P^J on irreducible components with deformed angular momentum J. Owing to (1.12), their matrix elements are connected with Clebsch-Gordan coefficients by the expression

$$P^J_{m_1 m_2, n_1 n_2} = \langle j m_1 j m_2 | P^J | j n_1 j n_2 \rangle = C^{JM}_{j m_1 j m_2} C^{JM}_{j n_1 j n_2}. \tag{2.3}$$

On the other hand, they are simply related to the Casimir operator. In fact

$$P^J = \left(\prod_{i=0, i \neq J}^{2j} (C - c_i) \right) \left(\prod_{i=0, i \neq J}^{2j} (c_J - c_i) \right)^{-1}. \tag{2.4}$$

It is evident from construction that

$$P^{J'} | JM \rangle = \delta_{JJ'} | JM \rangle. \tag{2.5}$$

From this relation or from the characteristic polynomial it follows that

$$P^J \cdot P^{J'} = \delta_{JJ'} P^J. \tag{2.6}$$

We can now write the decomposition of the interaction in terms of projectors:

$$I = \sum_{J=0}^{2j} \lambda_J P^J. \tag{2.7}$$

In this representation the coupling constants λ_J are simultaneously eigenvalues of the interaction. The elements of the basis are all polynomials of order $2j$ in the Casimir operator.

3. Braid groups, the Temperley–Lieb and the Hecke algebras

We are now interested in finding particular interactions whose local operators

$$I_k \qquad k = 1, \dots, n-1 \tag{3.1}$$

are elements of braid groups, the Hecke algebra or the Temperley-Lieb algebra.

A representation of braid groups is given if the operators I_k satisfy

$$\begin{aligned} I_k I_{k+1} I_k &= I_{k+1} I_k I_{k+1} \qquad && i = 1, \dots, n-2 \\ I_k I_l &= I_l I_k && |k - l| \geqslant 2. \end{aligned} \tag{3.2}$$

The Hecke algebra is given if in addition we have the condition

$$(I_k + 1)(I_k - q) = 0. \tag{3.3}$$

The Temperley-Lieb algebra is given by the relations

$$\begin{aligned} T_k T_{k \pm 1} T_k &= T_k \\ T_k^2 &= \beta T_k \\ T_k T_l &= T_l T_k \qquad |k - l| \geqslant 2. \end{aligned} \tag{3.4}$$

The Temperley–Lieb algebra is a special case of the braid group, which can be checked using the substitution

$$I_k = \sqrt{q}\, T_k - 1 \qquad \beta = q^{-1/2} + q^{1/2}. \tag{3.5}$$

Now we consider the interaction with the coupling constant:

$$\lambda_0 = [2j+1] \qquad \lambda_1 = \lambda_2 = \ldots \lambda_{2j} = 0. \tag{3.6}$$

This implies that only spins coupled to the total spin zero can interact. Explicitly

$$I = \pm[2j+1]P_0 \tag{3.7}$$

or

$$I = \frac{\pm(-1)^{2j}(C-[2])(C-[2][3]),\ldots,(C-[2j][2j+1])}{[2]^2[3]^2,\ldots,[2j]^2}. \tag{3.8}$$

We can show that this operator satisfies the Temperley–Lieb algebra (3.4) with

$$\beta = \pm[2j+1].$$

The second condition (3.4) is satisfied owing to the projector properties (2.6). To show the first condition (3.4), we calculate an explicit formula for the matrix element of I in the basis

$$\begin{aligned} &|m_1 m_2\rangle = |jm_1\rangle \times |jm_2\rangle \\ &I_{m_1 m_2, n_1 n_2} = \pm[2j+1]\langle m_1 m_2 | P_0 | n_1 n_2\rangle. \end{aligned} \tag{3.9}$$

Owing to (2.3), it is also

$$I_{m_1 m_2, n_1 n_2} = \pm[2j+1] C^{00}_{m_1 m_2} C^{00}_{n_1 n_2}. \tag{3.10}$$

From [6] we have

$$C^{00}_{m_1 m_2} = (-1)^{j-m_1} \delta_{m_1+m_2,0} \frac{q^{m_1}}{\sqrt{[2j+1]}} \tag{3.11a}$$

thus

$$I_{m_1 m_2, n_1 n_2} = \pm(-1)^{m_1-n_1} \delta_{m_1+m_2,0} \delta_{n_1+n_2,0} q^{m_1+n_1}. \tag{3.11b}$$

Now the Temperley–Lieb condition can be explicitly checked. The explicit matrix (3.11*b*) was written down by Batchelor *et al* [5], but now we understand its relation to the $SU(2)_q$-invariant Hamiltonian (owing to (3.6), (2.7) and (2.5)).

Now we shall show another interaction which satisfies braid-group requirements. It is well known that a solution I of the braid condition can be obtained from a solution R of the Yang–Baxter equation. The connection is

$$I = PR \qquad I_k = P_{k\,k+1} R_{k\,k+1} \tag{3.12}$$

where P permutes vectors defined in tensor products, e.g.

$$P_{12} |jm_1\rangle \times |jm_2\rangle = |jm_2\rangle \times |jm_1\rangle.$$

R in (3.12) satisfies the Yang–Baxter relation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{3.13}$$

and also

$$PR_{kl}P^{-1}=R_{P(k)P(p)}$$

where $P(k)$ denotes the integer to which the permutation P carries the integer k.

The check is straightforward. We can use the R matrix from the theory of q-deformed algebras, which is well known to satisfy the Yang–Baxter equation. This matrix is explicitly known [4, 7, 8], hence we can write the corresponding I with an arbitrary constant α:

$$I=\alpha\sum_J(-1)^J q^{J(J+1)}P_J. \tag{3.14}$$

Comparing (3.14) with (2.7), we find that our Hamiltonian satisfies braid relations for

$$\lambda_J=\alpha(-1)^J q^{J(J+1)}. \tag{3.15}$$

The Hecke relation is not satisfied for $q\neq 1$ because all eigenvalues are different, contrary to the implications of (3.3). It is interesting to note that for $q=1$ our Hamiltonian has the simple form

$$I=\alpha\sum_J(-1)^J P_J \tag{3.16}$$

and for $\alpha=\pm 1$ we have the Hecke algebra. For spin 1, these classical results coincide with the result of Batchelor *et al* [5]. For spin $\frac{3}{2}$ this Hamiltonian does not fall into the class claimed to be integrable by Chubukov and Khveschenko [9]. This Hamiltonian has a particularly simple matrix form

$$\begin{aligned} I_{m_1m_2,n_1n_2}&=\pm\sum_J(-1)^J(P_J)_{m_1m_2,n_1n_2}\\ &=\pm\sum_J(-1)^J C^{JM}_{jm_1jm_2}C^{JM}_{jn_1jn_2}. \end{aligned}$$

Owing to the symmetry property

$$C^{JM}_{jm_1jm_2}=C^{JM}_{jm_2jm_1}(-1)^{2j-J}$$

and to the orthogonality property of Clebsch–Gordan coefficients

$$I_{m_1m_2,n_1n_2}=\alpha(-1)^{2j}\delta_{m_1n_2}\delta_{m_2n_1}\qquad \alpha=\pm 1. \tag{3.17}$$

This relation is the explicit matrix form for the Hamiltonian which satisfies the Hecke algebra for arbitrary spin ($q=1$).

4. Higher symmetry

An interesting question is the relation of integrable points and higher symmetries. Consider the Temperley–Lieb point first. Here all non-singlet states are degenerate and thus it corresponds to the embedding

$$\mathrm{SU}(2)\subset\mathrm{SU}(2j+1). \tag{4.1}$$

Of two spins which form the interaction, one is represented in the V_{2j+1} or the fundamental representation of $\mathrm{SU}(2j+1)$, and the other in its complex conjugate $\bar{V}_{2j+1}$. Of course,

$$V_{2j+1}\times\bar{V}_{2j+1}=V_1\dot{+}V_{j^2-1} \tag{4.2}$$

in agreement with the degeneracy of the Temperley–Lieb interaction. For spin 1 it coincides with a case remarked on by Batchelor *et al* [5].

Now we consider the braid solution of (3.15).

All eigenvalues for $q\neq 1$ are different, so no embedding is possible. This is an interesting situation where we have an integrable point but no higher symmetry. Of course, an exception is the classical case

$$q=1 \qquad \lambda_J=\alpha(-1)^J. \tag{4.3}$$

For general α it satisfies the braid group and for $\alpha=\pm 1$, the Hecke algebra.

In both cases it corresponds to the embedding

$$\mathrm{SU}(2)\subset\mathrm{SU}(2j+1).$$

For this case both spin operators in the interaction are realized in the fundamental representation V_{2j+1}. Owing to

$$V_{2j+1}\times V_{2j+1}=V_{j(2j+1)}+V_{(j+1)(2j+1)} \tag{4.4}$$

we explain the symmetry content of the degeneracy of even and odd total spins.

This case is interesting for two reasons. For the quantum case we have an example of an integrable point which does not correspond to a higher symmetry. On the other hand, the classical integrable Hamiltonian corresponds to a higher symmetry, contrary to its quantum generalization.

5. Spin $\frac{3}{2}$

Thus far we have treated the questions for general spin. Now we want to explore spin $\frac{3}{2}$ in more detail. We shall present two explicit representations of the $\mathrm{SU}(2)_q$-invariant interaction: one in a matrix basis and the other in terms of spin operators. According to equation (2.2) a possible representation for the interaction operator is

$$I=\alpha_0 1+\alpha_1(C-c_3)+\alpha_2(C-c_3)(C-c_2)+\alpha_3(C-c_3)(C-c_2)(C-c_1). \tag{5.1}$$

It is useful to introduce a new basis in the space of invariants:

$$\begin{aligned}
I&=\beta_0 1+\beta_1 L_1+\beta_2 L_2+\beta_3 L_3\\
L_1&=\frac{3}{2[3]}C+\tfrac{9}{4}1\\
L_2&=\frac{3}{[3][2]^2}(C-c_3)(C-c_2)+\tfrac{27}{16}1\\
L_3&=\frac{1}{[2]^2[3]^2}(C-c_3)(C-c_2)(C-c_1)+\tfrac{33}{32}1.
\end{aligned} \tag{5.2}$$

The operators L_1, L_2 and L_3 can be considered as 16×16 matrices in the space

$$|jm_1\rangle \times |jm_2\rangle \qquad j=\tfrac{3}{2} \qquad m_1, m_2=\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}.$$

These matrices can be expressed for every q in terms of the following 19 numerical matrices (q-independent). There are ten diagonal matrices D_1, D_2, D_3, D_4, D_5, D_6, A_2, A_3, A_4, A_6. The elements of the diagonal are as follows (for clarity, we separate four blocks):

$$\begin{array}{lrrrr}
D_1: & 1000 & 0000 & 0000 & 0001 \\
D_2: & 0100 & 1000 & 0001 & 0010 \\
D_3: & 0010 & 0001 & 1000 & 0100 \\
D_4: & 0001 & 0000 & 0000 & 1000 \\
D_5: & 0000 & 0100 & 0010 & 0000 \\
D_6: & 0000 & 0010 & 0100 & 0000 \\
A_2: & 0100 & -1000 & 0001 & 00-10 \\
A_3: & 0010 & 0001 & -1000 & 0-100 \\
A_4: & 0001 & 0000 & 0000 & -1000 \\
A_6: & 0000 & 0010 & 0-100 & 0000.
\end{array} \tag{5.3}$$

Now we describe the non-diagonal matrices. One group contains M_1, M_2, M_3, M_4, B_2, B_3. The common feature of these matrices is that non-vanishing block diagonals are nearest to the left and to the right of the main diagonal. In these blocks non-zero diagonals are nearest to the main diagonal when looked from the inner side of the matrix. For example, M_1 is

$$\begin{bmatrix}
 & & & & 0 & & & & & & & & & & & \\
 & & & & 1 & 0 & & & & & & & & & & \\
 & & 0 & & & 0 & 0 & & & 0 & & & & & 0 & \\
 & & & & & & 0 & 0 & & & & & & & & \\
0 & 1 & & & & & & & 0 & & & & & & & \\
 & 0 & 0 & & & 0 & & & 0 & 0 & & & & & 0 & \\
 & & 0 & 0 & & & & & & 0 & 0 & & & & & \\
 & & & 0 & & & & & & & 0 & 0 & & & & \\
 & & & & 0 & 0 & & & & & & & 0 & & & \\
 & & & & & 0 & 0 & & & & & & 0 & 0 & & \\
 & & 0 & & & & 0 & 0 & & 0 & & & & 1 & 0 & \\
 & & & & & & & 0 & & & & & & & 0 & 0 \\
 & & & & & & & & 0 & 0 & & & & & & \\
 & & 0 & & & 0 & & & & 0 & 1 & & & & 0 & \\
 & & & & & & & & & & 0 & 0 & & & & \\
 & & & & & & & & & & & 0 & & & &
\end{bmatrix}. \tag{5.4}$$

Symbolically,

$$M_1: \quad 100 \quad 000 \quad 010.$$

Similarly,

$$\begin{array}{llll} M_2: & 010 & 101 & 010 \\ M_3: & 001 & 000 & 100 \\ M_4: & 000 & 010 & 000 \\ B_2: & 010 & -101 & 0-10 \\ B_3: & 001 & 000 & -100. \end{array} \tag{5.5}$$

The next group of matrices are N_1, N_2 and B_4. In the description of the preceding group one has to replace the block diagonal first left and first right with the second left and second right. Similarly, in these blocks, the diagonal nearest to the main diagonal has to be replaced with the second nearest.

Symbolically,

$$\begin{array}{lll} N_1: & 10 & 01 \\ N_2: & 01 & 10 \\ B_4: & 01 & -10. \end{array} \tag{5.6}$$

Finally, there is the matrix P. Its only non-vanishing elements are

$$P_{1,16} = P_{16,1} = 1. \tag{5.7}$$

Now we can give an explicit expression of the invariants L_1, L_2 and L_3 in their matrix basis:

$$\begin{aligned} L_1 = SS + \tfrac{3}{2}\Bigg[& 2\left(\sinh\frac{3\lambda}{2}\right)^2 [D_1 - D_5 - D_6] - 2\left(\sinh\frac{\lambda}{2}\right)^2 [D_3 + 2D_4] \\ & + \left(\frac{\cosh\lambda}{[3]} - \frac{1}{3}\right)[2D_5 + D_6] - \sinh 3\lambda [A_2 + A_3 + A_4] \\ & - \sinh\lambda [A_3 + 2A_6] - \frac{\sinh\lambda}{[3]} A_6 - \frac{\sinh 2\lambda}{\sqrt{[3]}} B_2 - \sinh 2\lambda B_3 \\ & + 2\left(\frac{\cosh^2}{\sqrt{[3]}} - \frac{1}{\sqrt{3}}\right) M_2 + 2\sinh^2\lambda\, M_3 - \frac{4\sinh^2\lambda}{3[3]} M_4 \Bigg] \end{aligned}$$

$$\begin{aligned} L_2 = (SS)^2 - \tfrac{3}{2}SS & + 6(\sinh 2\lambda)^2 D_3 + 6[(\sinh 2\lambda)^2 + (\sinh\lambda)^2] D_4 \\ & - \frac{4(\sinh\lambda)^2}{[3]} D_5 + \left(6(\sinh 2\lambda)^2 - \frac{2(\sinh\lambda)^2}{[3]}\right) D_6 \\ & + 3\sinh 4\lambda\, A_3 + 3(\sinh 4\lambda + \sinh 2\lambda) A_4 \\ & + 3\left(\sinh 4\lambda - 2\sinh 2\lambda - \frac{2\sinh 2\lambda}{[3]}\right) A_6 \\ & + 2\sqrt{3}\left[1 - \cosh\lambda\cosh 2\lambda \left(\frac{3}{[3]}\right)^{1/2}\right] M_2 - 12\left(\sinh\frac{\lambda}{2}\right)^2 M_3 \\ & + \left[-12\left(\sinh\frac{3\lambda}{2}\right)^2 + \frac{6\cosh\lambda}{[3]} - 2\right] M_4 - \frac{3(\sinh 3\lambda + \sinh\lambda)}{\sqrt{[3]}} B_2 \\ & - 3\sinh 2\lambda\, B_4 + 6(\sinh\lambda)^2 N_2 \end{aligned} \tag{5.8}$$

$$L_3=\tfrac{2}{9}(\boldsymbol{SS})^3+\tfrac{5}{18}(\boldsymbol{SS})^2-\tfrac{31}{24}\boldsymbol{SS}-2\left(\sinh\frac{3\lambda}{2}\right)^2 D_4$$
$$-2\left(\sinh\frac{\lambda}{2}\right)^2 D_6-\sinh 3\lambda\, A_4-\sinh\lambda\, A_6+2(\sinh\lambda)^2 M_3$$
$$+\sinh 2\lambda\, B_3-2\left(\sinh\frac{\lambda}{2}\right)^2 N_2-\sinh\lambda\, B_4.$$

Here $\boldsymbol{S}$ is the classical spin-$\frac{3}{2}$ operator and the parameter λ is related to q from (1.2) by

$$q=\mathrm{e}^{\lambda}.$$

Physical models are usually expressed in terms of the usual SU(2) spin operators. For this purpose we present the relation of our matrix basis to the basis with spin operators as elements. First we introduce the following operators:

$$\begin{aligned}
\Omega_{k,k+1}&=J_k^xJ_{k+1}^x+J_k^yJ_{k+1}^y\\
\Sigma_{k,k+1}^{ij}&=\tfrac{1}{2}[(J_k^z)^i(J_{k+1}^z)^j+(J_k^z)^j(J_k^z)^i]\\
\Delta_{k,k+1}^{ij}&=\tfrac{1}{2}[(J_k^z)^i(J_{k+1}^z)^j-(J_k^z)^j(J_{k+1}^z)^i]\\
\Phi_{k,k+1}&=[(J_k^xJ_{k+1}^x+J_k^yJ_{k+1}^y)^2-(J_k^xJ_{k+1}^y-J_k^yJ_{k+1}^x)^2]
\end{aligned}\tag{5.9}$$

where $i,j=0,1,\ldots$ denote the power to which the spin operator is raised. For simplicity we shall omit the indices k, $k+1$ and write

$$\begin{aligned}
\Omega_{k,k+1}&=\Omega &\qquad \Sigma_{k,k+1}^{ij}&=\Sigma^{ij}\\
\Delta_{k,k+1}^{ij}&=\Delta^{ij} &\qquad \Phi_{k,k+1}&=\Phi.
\end{aligned}\tag{5.10}$$

Second, using this notation we can decompose the matrix basis in terms of spin operators as follows:

$$D_K=\sum_{i,j}a_{ij}^K\Sigma^{i,j}\qquad K=1,\ldots,6$$
$$(i,j)=(0,0),(1,1),(2,2),(3,3),(0,2),(1,3)\tag{5.11}$$

	a_{00}	a_{11}	a_{22}	a_{33}	a_{02}	a_{13}
$K=1$	$\frac{1}{128}$	$\frac{1}{288}$	$\frac{1}{8}$	$\frac{1}{18}$	$-\frac{1}{16}$	$-\frac{1}{36}$
$K=2$	$-\frac{9}{64}$	$-\frac{3}{16}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$\frac{5}{8}$	$\frac{5}{6}$
$K=3$	$-\frac{9}{64}$	$\frac{3}{16}$	$-\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{8}$	$-\frac{5}{6}$
$K=4$	$\frac{1}{128}$	$-\frac{1}{288}$	$\frac{1}{8}$	$-\frac{1}{18}$	$-\frac{1}{16}$	$\frac{1}{36}$
$K=5$	$\frac{81}{128}$	$\frac{81}{32}$	$\frac{1}{8}$	$\frac{1}{2}$	$-\frac{9}{16}$	$-\frac{9}{4}$
$K=6$	$\frac{81}{128}$	$-\frac{81}{32}$	$\frac{1}{8}$	$-\frac{1}{2}$	$-\frac{9}{16}$	$\frac{9}{4}$

$$A_M=\sum_{i,j}a_{ij}^M\Delta^{ij}\qquad M=2,3,4,6\tag{5.12}$$

	a_{01}	a_{03}	a_{21}	a_{32}
$M=2$	$-\frac{3}{16}$	$-\frac{1}{4}$	$\frac{13}{12}$	$\frac{1}{3}$
$M=3$	$\frac{3}{8}$	$-\frac{1}{2}$	$-\frac{7}{6}$	$-\frac{2}{3}$
$M=4$	$-\frac{1}{96}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{6}$
$M=5$	$-\frac{81}{32}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{1}{2}$

$$M_A=\sum_{i,j}a_{ij}^A\{\Omega\Sigma^{ij}+\mathrm{HC}\}\tag{5.13}$$

	a_{00}	a_{02}	a_{11}	a_{22}
$A=1$	$\frac{3}{32}$	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$
$A=2$	$-\frac{9}{16}$	$\frac{3}{2}$	-1	-1
$A=3$	$\frac{1}{96}$	$-\frac{1}{12}$	0	$\frac{1}{6}$
$A=4$	$\frac{29}{64}$	$-\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{4}$

$$B_K = \sum_{i,j} a_{ij}^K \{\Omega\Delta^{ij} + \mathrm{HC}\} \qquad K = 2, 3 \tag{5.14}$$

	a_{10}	a_{12}
$K=2$	$\frac{7}{4}$	-1
$K=3$	$-\frac{1}{4}$	$\frac{1}{3}$

$$N_2 = \sum_{i,j} a_{ij}^2 \Phi\Sigma^{ij} + \mathrm{HC} \tag{5.15}$$

$$a_{00}^2 = -\tfrac{1}{8} \qquad a_{11}^2 = -\tfrac{1}{6}$$

$$B_4 = 4\Phi\Delta^{ij}. \tag{5.16}$$

Finally we mention the interesting question about the general solution of braid-group conditions. A systematic analysis of these conditions for spin $\frac{3}{2}$ shows that there are not other generic solutions besides those already mentioned. We do not exclude solutions for special values of q. Details will be given elsewhere.

6. Conclusion

We have presented a general and arbitrary spin Hamiltonian for an n-dimensional spin chain which is invariant under the $\mathrm{SU}(2)_q$ algebra. For two special values of coupling constants local interactions satisfy the Temperley–Lieb algebra. In addition, another one-parametric family of interactions that satisfies braid-group requirements has been found. For the classical value of q $(q=1)$ and for two values of coupling constants we obtain Hecke relations. It is interesting to note that for $q \neq 1$ such points do not always correspond to a higher symmetry.

We have considered the case of spin $\frac{3}{2}$ in more detail. Two additional representations of the Hamiltonian have been written: one is in terms of matrix basis and the other in terms of spin operators ('physical basis'). For spin $\frac{3}{2}$ we have found that solutions presented in the first part of the paper are in fact general solutions to braid-group requirements in the generic sense.

Acknowledgments

We would like to thank Professor V Rittenberg for frequent discussions and suggestions. SP is grateful to the Physikalisches Institut der Universität Bonn, the Dipartimento di Fisica dell'Università 'La Sapienza' and the INFN, Roma for kind hospitality. SP and SM would like to acknowledge the financial support of the NSF Grant 683. SP, SM and MM wish to thank the Science Foundation of the R Croatia, SFR Yugoslavia and the Alpe-Adria Working community research projects for financial support.

References

[1] Baxter R J 1982 *Exactly Solvable Models in Statistical Mechanics* (New York: Academic)

[2] Barber M N and Batchelor M T 1989 *Phys. Rev.* B **40** 4621
Papanicolaou N 1968 *Nucl. Phys.* B **305** 367

[3] Ginsparg P 1989 Some statistical mechanical models and conformal field theories *Lectures (Trieste Spring School, April 3-11, 1989)*

[4] Drinfeld V G 1987 Quantum Groups Proc. Int. Congr. of Mathematicians, Berkeley, 1986 (Providence, RI: American Mathematical Society) pp 798-820
Kirillov A N and Reshetikhin N Yu 1989 *Advanced Series in Mathematical Physics* vol 7, ed V G Kac (Singapore: World Scientific) p 285
Majid S 1990 *Int. J. Mod. Phys.* A **5** 1

[5] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A: Math. Gen.* **23** L141

[6] Nomura M 1989 *J. Math. Phys.* **30** 2397

[7] Jimbo M 1989 *Int. J. Mod. Phys.* A **15** 3759

[8] Pasquier V 1988 *Commun. Math. Phys.* **118** 355

[9] Chubukov A V and Khveschenko D V 1987 *J. Phys. C: Solid State Phys.* **20** L505
Jian Min Liu and Bonner J C 1988 *J. Appl. Phys.* **63** 4144